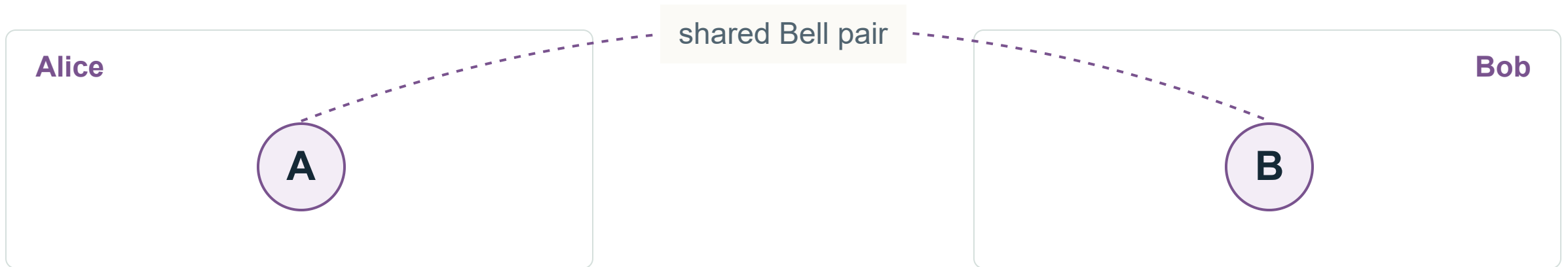


What can a shared entangled pair help us send?

Alice and Bob are in different places. Each holds one qubit from a Bell pair.



Could this help Alice send an unknown qubit state?
Could it help her send a classical message?

Entanglement is a resource we can use for communication.

- 01 A Bell pair behaves differently from a shared random bit.
 - 02 With a shared Bell pair, two classical bits transfer an unknown qubit state.
 - 03 With a shared Bell pair, one transmitted qubit carries two classical bits.
-

A Bell pair has one joint state across two places.



$$|\Phi^+\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

A is Alice's qubit; B is Bob's. Every two-qubit ket uses the order A, B.

The pair is in a superposition of 00 and 11.

Matching readouts do not establish that a pair is entangled.

BELL PAIR

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The same superposition on every run.

SHARED RANDOM BIT

A fair random choice on each run:

we prepare either $|00\rangle$ or $|11\rangle$.

Direct measurement	00	01	10	11
Either preparation	$\frac{1}{2}$	0	0	$\frac{1}{2}$

This readout alone cannot distinguish the two preparations.

The 01 and 10 contributions cancel for the Bell pair.

Alice and Bob each apply H to their qubit.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{H \otimes H} \frac{|++\rangle + |--\rangle}{\sqrt{2}}.$$

We expand the two terms with their original coefficients:

$$\frac{|++\rangle}{\sqrt{2}} = \frac{1}{2\sqrt{2}} (|00\rangle + |01\rangle + |10\rangle + |11\rangle),$$

$$\frac{|--\rangle}{\sqrt{2}} = \frac{1}{2\sqrt{2}} (|00\rangle - |01\rangle - |10\rangle + |11\rangle).$$

$$\frac{|++\rangle + |--\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The 01 and 10 terms cancel; the 00 and 11 terms add. The Bell state is unchanged.

Hadamards distinguish the Bell pair from the shared random bit.

The Bell pair returns to $(|00\rangle + |11\rangle)/\sqrt{2}$.

The random preparation becomes either $|++\rangle$ or $--\rangle$, on separate runs.

$$|++\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}.$$

Will their results still always agree in both cases?

After H on both	00	01	10	11
Bell pair	$\frac{1}{2}$	0	0	$\frac{1}{2}$
Shared random bit	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

The Bell pair always agrees. The random preparation agrees only half the time.

A shared Bell pair alone cannot carry Alice's message.

Alice wants to send Bob one bit using only their shared pair.

They agree beforehand: 0 means no gate; 1 means X on A.

Intended bit → action	Joint state of A, B	Bob: 0	Bob: 1
0 → no gate	$(00\rangle + 11\rangle)/\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1 → X on A	$(01\rangle + 10\rangle)/\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Bob knows the rule. His own records do not reveal Alice's choice.

Adding H before Bob's measurement still gives 0 or 1 with probability $\frac{1}{2}$ each.

Bob needs a qubit state produced by Alice's computation.

Alice has a qubit Q in an unknown state:

$$|\psi\rangle_Q = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

ALICE

Q holds the state to be transferred.

She is not given α and β .

BOB

His next circuit needs a qubit in the same state.

He does not need its classical description.

Can a pair shared earlier help us do this using classical messages?

We cannot extract or perfectly copy an arbitrary unknown qubit state.

READING THE INPUT

$$\alpha|0\rangle + \beta|1\rangle$$

One measurement gives an outcome.

It does not give a complete description of α and β .

TRYING TO COPY IT

Any linear operation that copies 0 and 1:

$$|0\rangle|0\rangle \mapsto |00\rangle, \quad |1\rangle|0\rangle \mapsto |11\rangle$$

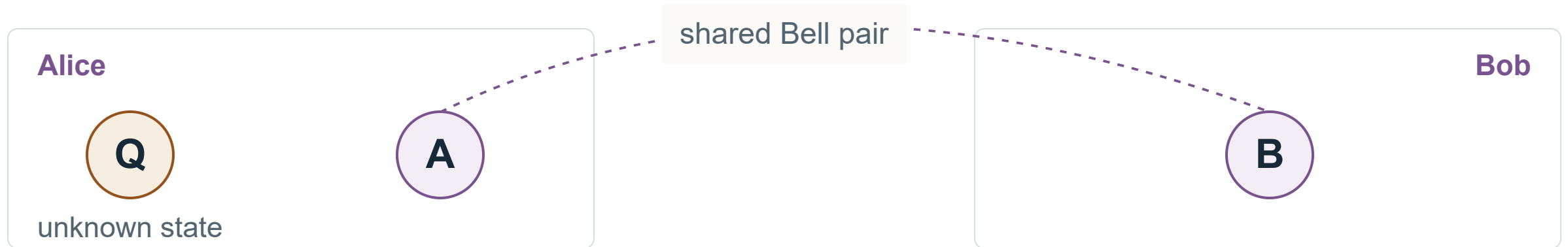
must act on plus as:

$$|+\rangle|0\rangle \mapsto \frac{|00\rangle + |11\rangle}{\sqrt{2}} \neq |+\rangle|+\rangle.$$

The state has to be transferred without first learning its amplitudes.

Teleportation transfers an unknown qubit state using a shared pair and two classical bits.

Q is the input. A and B form a fresh Bell pair shared earlier.



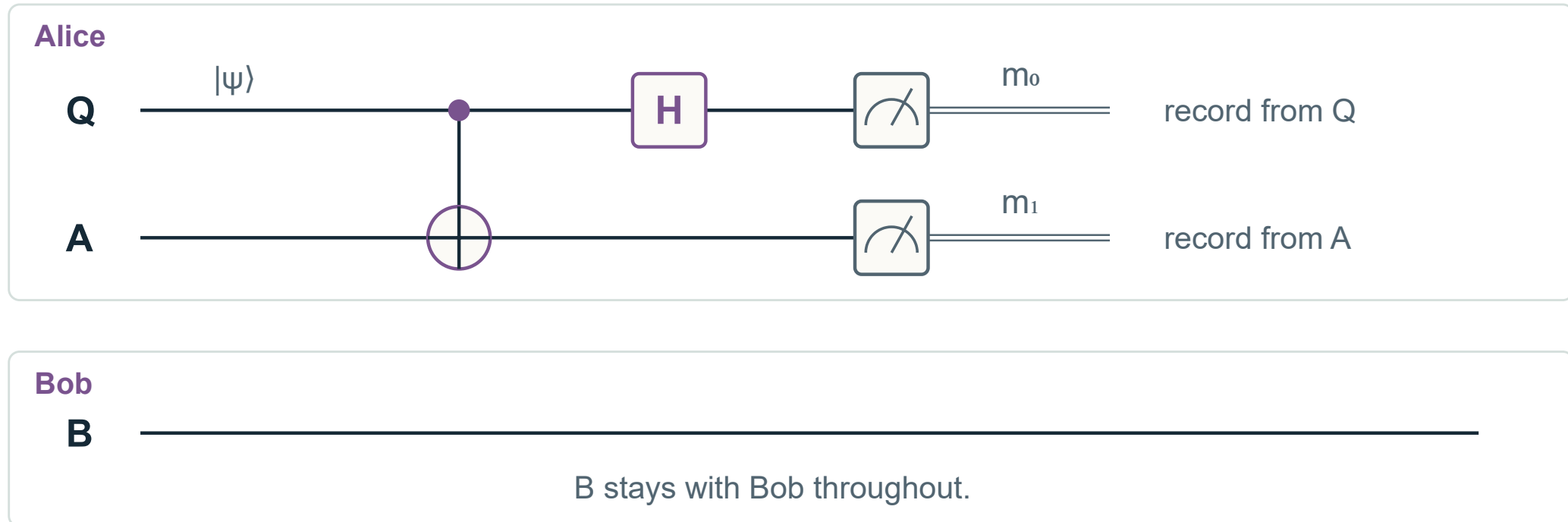
Already shared: one Bell pair A, B.

Sent during teleportation: two classical measurement bits.

Result: B acquires the input state $|\psi\rangle$.

Three-qubit kets use the order Q, A, B.

Alice couples the input to her Bell half before measuring both.



Q controls a CNOT on A. Then H acts on Q.

Q starts in $|\psi\rangle$. A and B start jointly in $|\Phi^+\rangle$.

m_0 is the result from Q; m_1 is the result from A.

The CNOT flips A in the terms where Q is one.

$$|\psi\rangle_Q |\Phi^+\rangle_{AB} = (\alpha|0\rangle + \beta|1\rangle) \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

We expand the product. Every term below has the common factor $1/\sqrt{2}$.

Before CNOT	After CNOT from Q to A
$\alpha 000\rangle$	$\alpha 000\rangle$
$\alpha 011\rangle$	$\alpha 011\rangle$
$\beta 100\rangle$	$\beta 110\rangle$
$\beta 111\rangle$	$\beta 101\rangle$

Order: Q, A, B. Only the middle label flips, and only in the last two rows.

Result 01 exchanges the input amplitudes on Bob's qubit.

H acts only on Q, so A stays unchanged. Result 01 requires A = 1.

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Term before H	Contribution with Q, A = 0, 1
$\frac{\alpha}{\sqrt{2}} 011\rangle$	$\frac{\alpha}{2} 01\rangle_{QA} 1\rangle_B$
$\frac{\beta}{\sqrt{2}} 110\rangle$	$\frac{\beta}{2} 01\rangle_{QA} 0\rangle_B$

$$\frac{1}{2}|01\rangle_{QA}(\beta|0\rangle + \alpha|1\rangle)_B = \frac{1}{2}|01\rangle_{QA}X|\psi\rangle_B.$$

If Alice measures 01, Bob's normalized state is $X|\psi\rangle$.

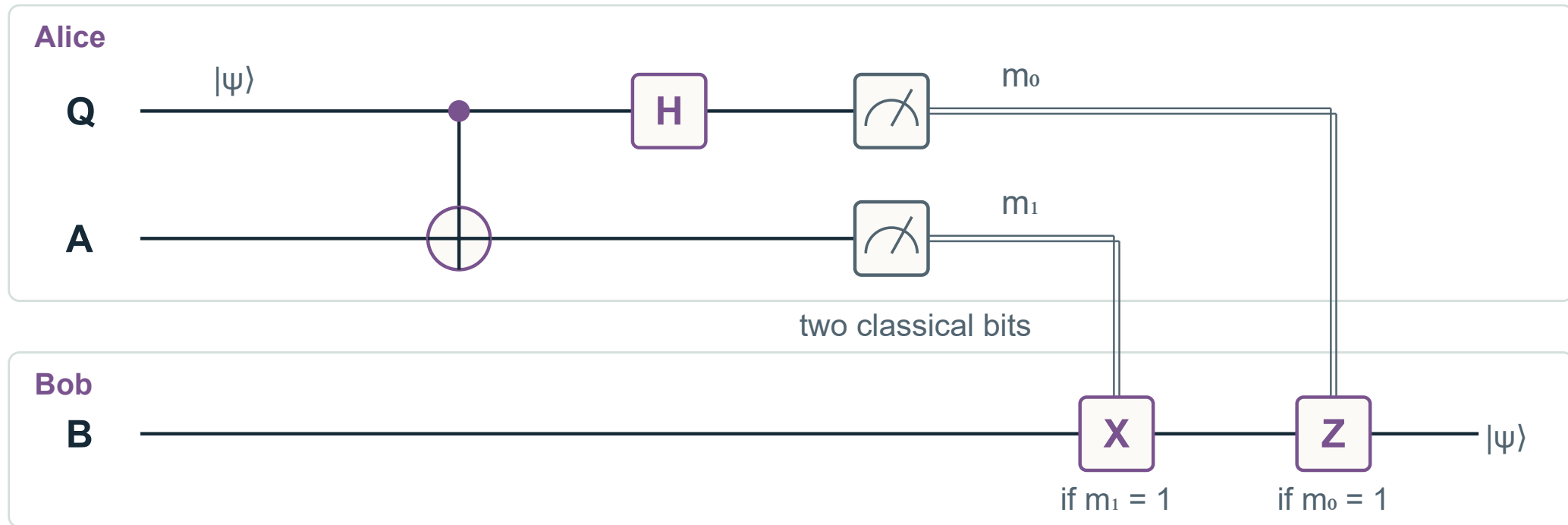
Each of Alice's four results gives Bob a known correction.

The same calculation for the other results gives:

m_0m_1	Bob's state, for that result	Correction, in time order
00	$\alpha 0\rangle + \beta 1\rangle$	No gate
01	$\beta 0\rangle + \alpha 1\rangle$	X
10	$\alpha 0\rangle - \beta 1\rangle$	Z
11	$-\beta 0\rangle + \alpha 1\rangle$	X, then Z

After the correction, B is in $\alpha|0\rangle + \beta|1\rangle$ in every case.

The two classical bits tell Bob which gates to apply.



$m_1 = 1$: we apply X. Then, if $m_0 = 1$, we apply Z.

A and B were entangled beforehand. The double lines carry Alice's measurement bits to Bob.

Alice's two bits select a correction without revealing the input state.

00

$\frac{1}{4}$

01

$\frac{1}{4}$

10

$\frac{1}{4}$

11

$\frac{1}{4}$

For example, the probability of Alice's result 01 is:

$$P(01) = \left| \frac{\alpha}{2} \right|^2 + \left| \frac{\beta}{2} \right|^2 = \frac{1}{4}.$$

All four results have probability $\frac{1}{4}$, for every input $|\psi\rangle$.

The result names a correction. It is not a classical description of the unknown amplitudes.

Bob receives the state, and Alice retains no second copy.

ALICE

Q and A have been measured.

She retains the classical results m_0, m_1 .

BOB, AFTER CORRECTION

$$|\psi\rangle_B = \alpha|0\rangle + \beta|1\rangle$$

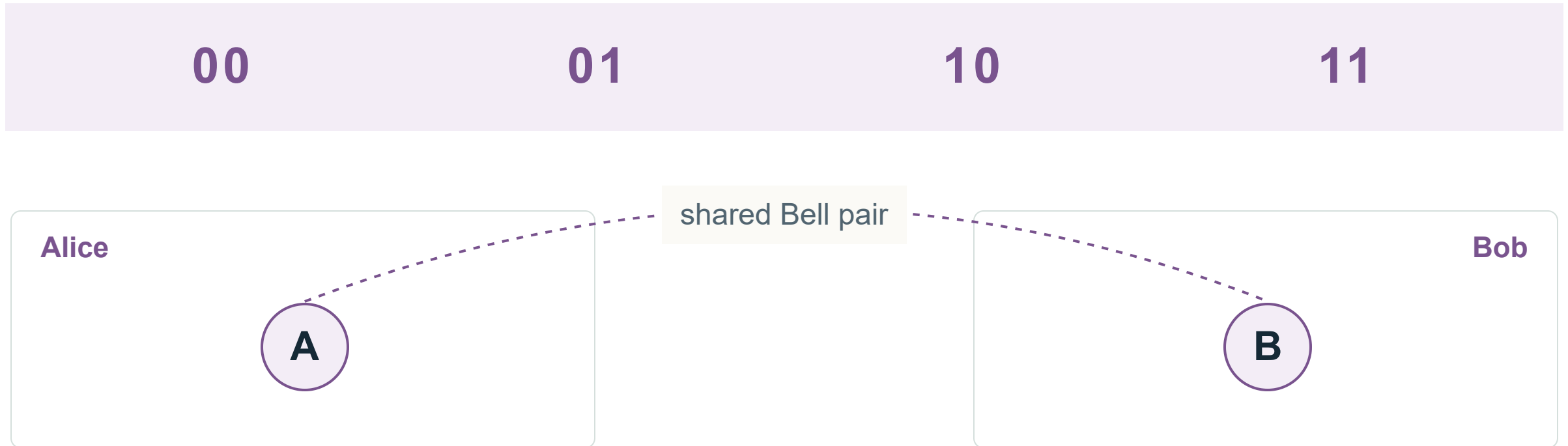
He can use B as the input to a further circuit.

The shared Bell pair has been used up.

The classical bits must arrive before Bob can recover an arbitrary input state.

A shared Bell pair lets Alice send two classical bits with one qubit.

The message is now classical. Alice chooses one of four bit pairs.



Superdense coding: Alice encodes the message on A, sends A, and Bob decodes using A and B.

Z changes the Bell pair but leaves the random preparation unchanged.

We compare message 00 (no gate) with message 10 (Z on A).

BELL PAIR

Before: $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

After Z: $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$

A relative sign changes.

Direct readout probabilities stay the same.

SHARED RANDOM BIT

Each run starts in 00 or 11.

$$|00\rangle \xrightarrow{Z \text{ on A}} |00\rangle,$$

$$|11\rangle \xrightarrow{Z \text{ on A}} -|11\rangle.$$

An overall sign leaves each prepared state unchanged.

With the random preparation, messages 00 and 10 produce the same state and cannot be distinguished.

Alice's four gate choices produce four Bell states.

Every row starts from $(|00\rangle + |11\rangle)/\sqrt{2}$.

Message	Gates on A in time order	Joint state after encoding
00	No gate	$(00\rangle + 11\rangle)/\sqrt{2}$
01	X	$(01\rangle + 10\rangle)/\sqrt{2}$
10	Z	$(00\rangle - 11\rangle)/\sqrt{2}$
11	X, then Z	$(01\rangle - 10\rangle)/\sqrt{2}$

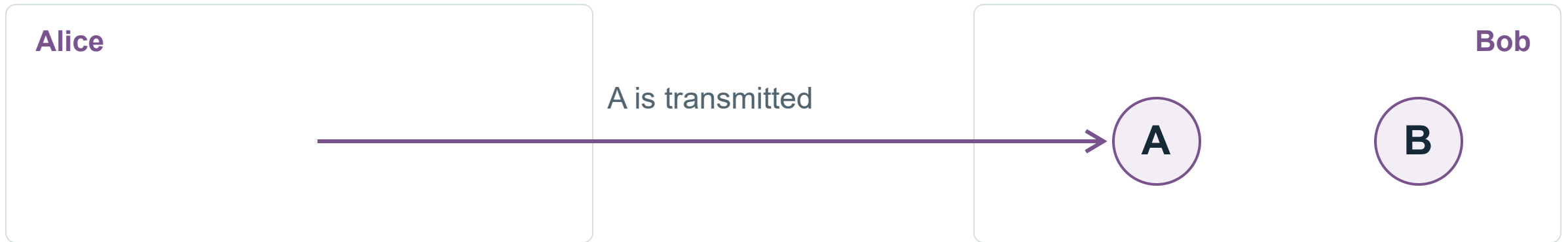
X changes which labels are paired. Z changes the relative sign.

Bob needs Alice's qubit to distinguish the four messages.

Before A arrives, Bob holds only B.

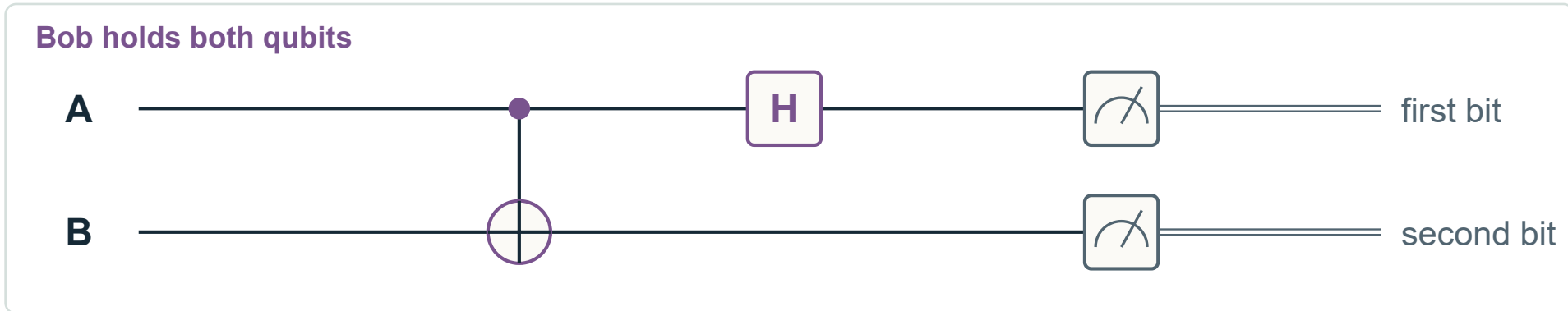
Can he already tell which of the four gates Alice chose?

No. His local measurement statistics are the same for all four messages.



After A arrives, Bob can apply a two-qubit decoder to A and B.

Bob's CNOT and H turn the relative sign into a measured bit.



For the message 10:

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT } A \rightarrow B} \frac{|00\rangle - |10\rangle}{\sqrt{2}} = |-\rangle_A |0\rangle_B.$$

$$|-\rangle_A |0\rangle_B \xrightarrow{H \text{ on } A} |10\rangle.$$

The measured bits, in order A, B, reproduce Alice's message: 10.

The decoder maps the four Bell states to four distinct bit pairs.

Received state of A, B	After CNOT A → B	After H on A
$(00\rangle + 11\rangle)/\sqrt{2}$	$ +\rangle 0\rangle$	$ 00\rangle$
$(01\rangle + 10\rangle)/\sqrt{2}$	$ +\rangle 1\rangle$	$ 01\rangle$
$(00\rangle - 11\rangle)/\sqrt{2}$	$ -\rangle 0\rangle$	$ 10\rangle$
$(01\rangle - 10\rangle)/\sqrt{2}$	$ -\rangle 1\rangle$	$ 11\rangle$

CNOT → H → measurement also gave Alice her four teleportation outcomes.

The first output bit distinguishes the sign; the second distinguishes matching from opposite labels.

Shared entanglement doubles the classical bits per transmitted noiseless qubit.

WITHOUT SHARED ENTANGLEMENT

1 classical bit

per transmitted noiseless qubit

WITH ONE SHARED BELL PAIR

2 classical bits

per transmitted noiseless qubit

We count transmission **after the pair has been shared**.

Preparing, distributing and storing the pair also require resources. Each message uses up one shared pair.

ENTANGLEMENT AS A RESOURCE

Entanglement changes what a given transmission can accomplish.

The shared Bell pair is the resource in both protocols.

	Teleportation	Superdense coding
Shared beforehand	One Bell pair	One Bell pair
Sent during the protocol	Two classical bits	One qubit
Delivered to Bob	One unknown qubit state	Two classical bits

A shared random bit cannot replace this Bell pair. Entanglement alone does not transmit the message.

A CIRCUIT TO FOLLOW

How does the decoder recover the message 11?

We begin with $(|00\rangle + |11\rangle)/\sqrt{2}$, in A, B order.

- 01 Alice applies X, then Z, to A.
What is the joint state?
- 02 Alice sends A. Bob applies CNOT from A to B, then H on A.
What is the state after each gate, and what are the measured bits?
- 03 If A never reaches Bob, can his own qubit reveal the message?

Optional: if Alice applies Z then X, does the measured message change?