

Gates and interference

QUANTUM GATE

A reversible operation on one or more qubits that transforms their state.

QUANTUM CIRCUIT

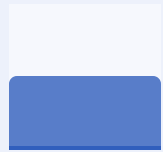
Specifies the operations, the qubits they act on and their order. Usually drawn as a diagram.

We use sequences of gates to carry out a quantum computation.

Can the same gate make these inputs distinguishable?

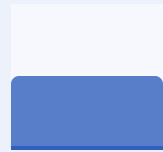
PLUS

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



0.5

0

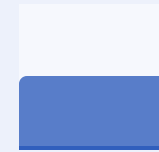


0.5

1

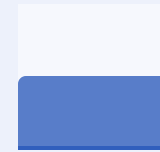
MINUS

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



0.5

0



0.5

1

Direct computational-basis readout: $P(0) = P(1) = 1/2$ for both inputs.

Yes. After the same gate, plus can give 0 with certainty and minus can give 1 with certainty.

A circuit's measurement probabilities depend on how its gates change and combine amplitudes.

- 01 Ideal gates transform a qubit's amplitudes reversibly.

- 02 A circuit fixes which operations act on each qubit and in what order.

- 03 Interference turns relative phase into different measurement probabilities.

A gate applies a linear transformation to the qubit's state.

$$U(\alpha|0\rangle + \beta|1\rangle) = \alpha U|0\rangle + \beta U|1\rangle$$

FOR EXAMPLE: X

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$$

SWAP THE AMPLITUDES

$$X \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

The same rule applies to every input.

The gate acts without reading α and β from the qubit.

Unitary gates preserve normalization and have inverses.

$$U^\dagger U = I$$

$U^\dagger$: transpose and complex conjugate.
Apply U , then $U^\dagger$: recover the input.

FOR EXAMPLE: Z

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Change the second amplitude's sign.

CHANGE THE SIGN TWICE

$$|+\rangle \xrightarrow{Z} |-\rangle \xrightarrow{Z} |+\rangle$$

$$Z^2 = I$$

The total probability stays 1. Individual probabilities can change.

Ideal single-qubit gates rotate the state point.

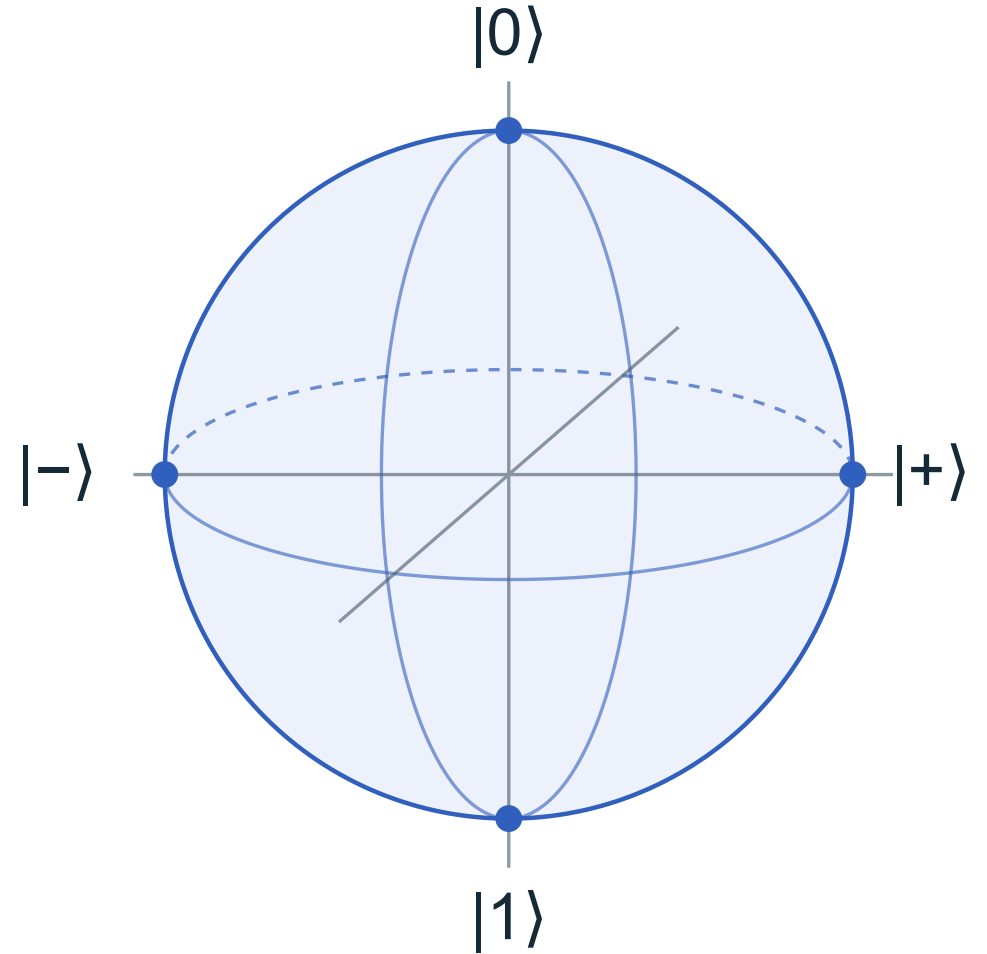
TWO EXAMPLES

$$X|0\rangle = |1\rangle$$

$$Z|+\rangle = |-\rangle$$

Gates can also have adjustable rotation angles.

An inverse gate reverses the ideal transformation.



The Hadamard gate combines the two input amplitudes.

Starting from 0, H gives us a way to prepare a superposition.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

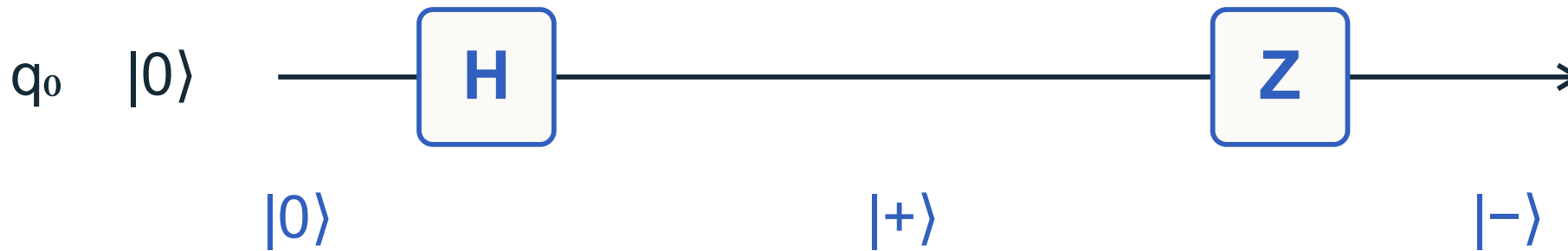
$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

First row: add. Second row: subtract. Scale both by $1/\sqrt{2}$.

From zero, H prepares the plus state on every run.

Each quantum wire follows one qubit through a sequence of operations.



Wire: the same qubit.

Box: an operation.

Time: left to right →

The output of one gate is the input to the next.

The intermediate states are written for our calculation. There is no measurement between these gates.

The input and measurement specify the calculation we need.



$$|0\rangle \xrightarrow{H} |+\rangle, \quad P(1) = \frac{1}{2}$$

q_0 : the qubit remains available.

c_0 : the classical record, 0 or 1.

The state afterwards matches the record: $|0\rangle$ or $|1\rangle$.

Our examples start in zero and request a final computational-basis readout.

Changing gate order can change the state we obtain.

A / H, THEN Z



$$|0\rangle \xrightarrow{H} |+\rangle \xrightarrow{Z} |-\rangle$$

$$ZH|0\rangle = Z(H|0\rangle)$$

B / Z, THEN H



$$|0\rangle \xrightarrow{Z} |0\rangle \xrightarrow{H} |+\rangle$$

$$HZ|0\rangle = H(Z|0\rangle)$$

Different states. The same direct readout probabilities: $P(0) = P(1) = 1/2$.

Keep the relative sign. A later operation can use it.

Which gate acts first?

$$ZH|0\rangle$$

H acts first. It is next to the input vector.

$$Z(H|0\rangle)$$

Z acts on the state H produces.

Next: make the phase difference affect the result we read.

Contributions to the same outcome can reinforce or cancel.

$$H \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

Each output amplitude receives contributions from both input amplitudes.



Add the contributions before calculating probabilities.

$$H|+\rangle = \frac{H|0\rangle + H|1\rangle}{\sqrt{2}}$$

FROM THE FIRST INPUT TERM

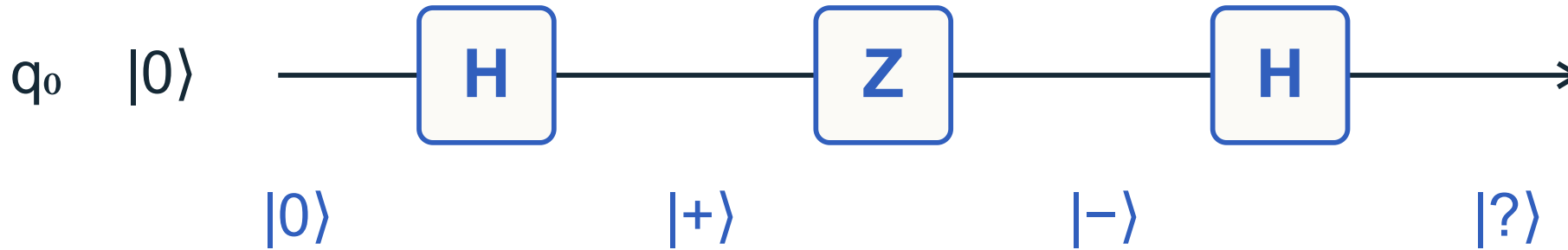
$$\frac{H|0\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{2}$$

FROM THE SECOND INPUT TERM

$$\frac{H|1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{2}$$

Outcome	Add contributions to its amplitude	Then take the squared magnitude
0	$\frac{1}{2} + \frac{1}{2} = 1$	$P(0) = 1 ^2 = 1$
1	$\frac{1}{2} - \frac{1}{2} = 0$	$P(1) = 0 ^2 = 0$

Z changes the relative sign before the last H.



BEFORE Z

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$P(0) = P(1) = 1/2$$

AFTER Z

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$P(0) = P(1) = 1/2$$

These are predicted readout probabilities. No measurement is inserted here.

A relative sign changes which contributions cancel.

$$H|-\rangle = \frac{H|0\rangle - H|1\rangle}{\sqrt{2}}$$

FROM THE FIRST INPUT TERM

$$\frac{H|0\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{2}$$

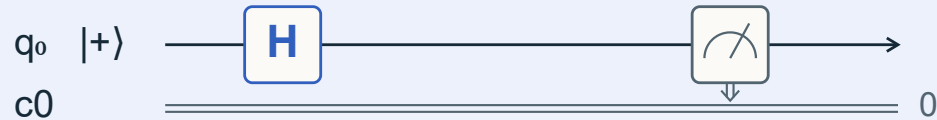
FROM THE SECOND INPUT TERM

$$-\frac{H|1\rangle}{\sqrt{2}} = \frac{-|0\rangle + |1\rangle}{2}$$

Outcome	Add contributions to its amplitude	Then take the squared magnitude
0	$\frac{1}{2} - \frac{1}{2} = 0$	$P(0) = 0 ^2 = 0$
1	$\frac{1}{2} + \frac{1}{2} = 1$	$P(1) = 1 ^2 = 1$

The same H gate makes plus and minus distinguishable.

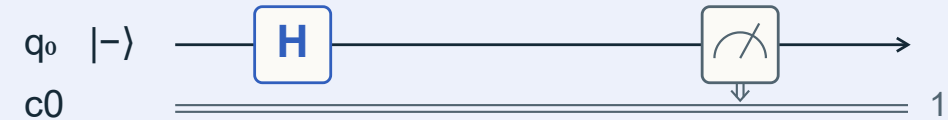
PLUS INPUT



$$H|+\rangle = |0\rangle$$

Record 0 with certainty.

MINUS INPUT



$$H|-\rangle = |1\rangle$$

Record 1 with certainty.

Relative phase can change the result we read.

H followed by this readout gives the plus/minus measurement probabilities.
In an algorithm, we arrange gates so that useful results become likely.

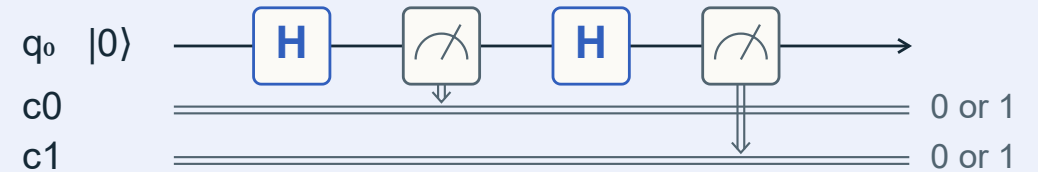
What happens if we measure between the two H gates?

WITHOUT THE MIDDLE MEASUREMENT



Final result: 0 with certainty.

WITH THE MIDDLE MEASUREMENT



What state enters the last H?

The middle measurement leaves $|0\rangle$ or $|1\rangle$, each with probability $1/2$.

The last H now receives one reference state or the other.

MIDDLE RECORD 0 / PROBABILITY $\frac{1}{2}$

$$|0\rangle \xrightarrow{H} |+\rangle$$

$$P(\text{final 1} \mid \text{middle 0}) = \frac{1}{2}$$

MIDDLE RECORD 1 / PROBABILITY $\frac{1}{2}$

$$|1\rangle \xrightarrow{H} |-\rangle$$

$$P(\text{final 1} \mid \text{middle 1}) = \frac{1}{2}$$

$$P(\text{final 1}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

Ignoring the middle record does not undo the state change.

Equal direct readout probabilities can hide different preparations.

Prepare a fresh input for each run.

Preparation	Direct readout: $P(1)$	H, then readout: $P(1)$
Prepare $ +\rangle$	$1/2$	0
Prepare $ -\rangle$	$1/2$	1
Randomly prepare $ 0\rangle$ or $ 1\rangle$, equally often	$1/2$	$1/2$

Amplitudes combine within a superposition.

¹⁹ Probabilities combine across randomly selected preparations.

A circuit's measurement probabilities depend on how its gates change and combine amplitudes.

- 01 Ideal gates transform a qubit's amplitudes reversibly.

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- 03 Interference turns relative phase into different measurement probabilities.

What is the probability of measuring 1?

Start in $|0\rangle$. Use a final computational-basis measurement.

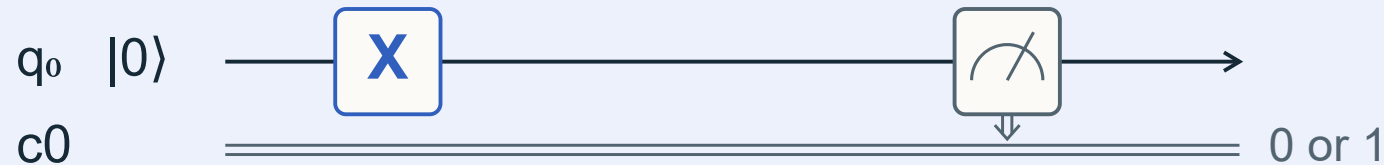
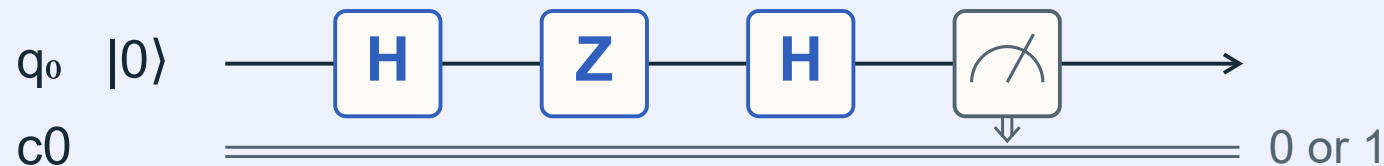


$$H|0\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$P(1) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} = \boxed{0.50}$$

Find the final probability of measuring 1 in each circuit.

Start in $|0\rangle$. Follow the state after each gate. Submit a decimal.

01**02****03**

Next: [Joint states and entanglement →](#)