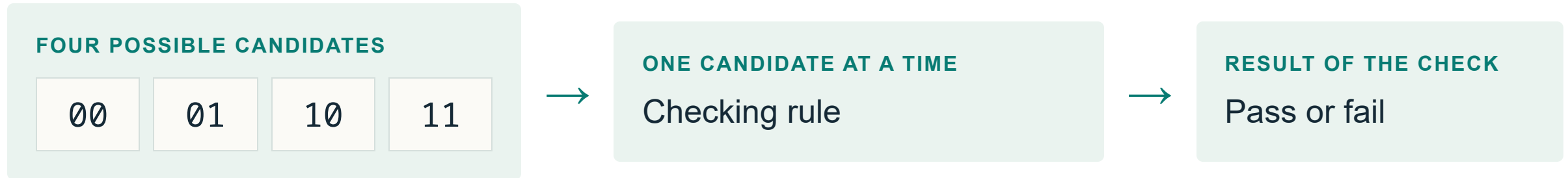


Grover search and amplitude amplification

We have a rule that checks a candidate.
We want to find a candidate that passes.



How many calls to the checking rule do we need?

$N = 4$ candidates; exactly one passes. We assume no additional structure that helps us locate it.
One call to the checking rule is a **query**.

Grover search uses interference to find a solution with fewer calls to a checking rule.

- 01 A phase mark preserves probabilities in the computational basis.
 - 02 Interference can make a marked solution more likely to appear.
 - 03 For one solution, Grover needs order $\sqrt{N}$ calls to the checking rule.
-

Our main case: one solution among N candidates, with no additional search structure.

The checking rule defines which candidates are solutions.

$f(x) = 1$: candidate x passes.

$f(x) = 0$: candidate x fails.

For the calculation, let's use a rule that accepts **10**.

An oracle implements access to this checking rule as a quantum operation.

Candidate labels use q_0q_1 . Candidate 10 means $q_0 = 1$ and $q_1 = 0$.

The accepted label is supplied so we can follow the arithmetic.

Candidate x	$f(x)$
00	0
01	0
10	1
11	0

Two Hadamards give the four candidates equal amplitudes.



$$|s\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

00

Amplitude $\frac{1}{2}$

01

Amplitude $\frac{1}{2}$

10

Amplitude $\frac{1}{2}$

11

Amplitude $\frac{1}{2}$

The phase oracle reverses the accepted candidate's amplitude.

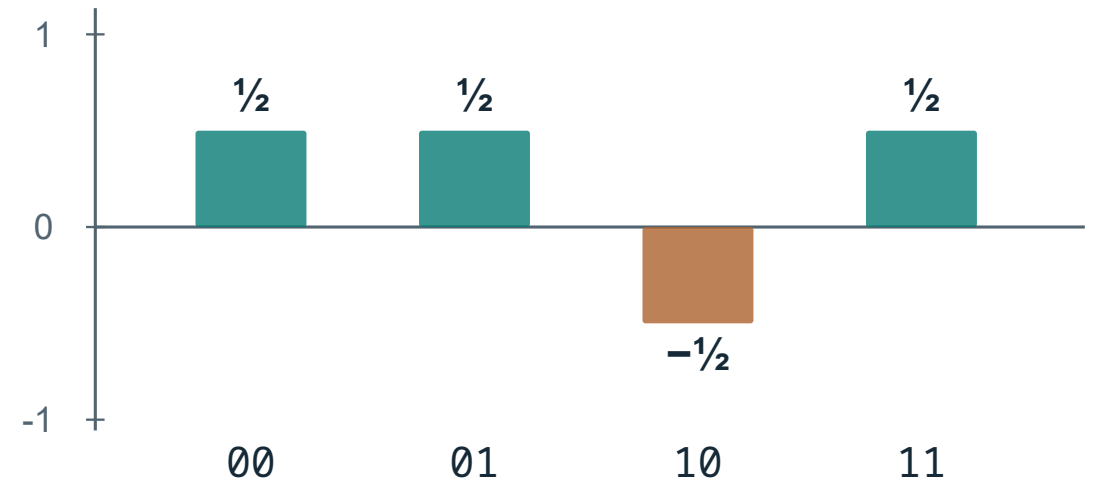
$$O_f|x\rangle = (-1)^{f(x)}|x\rangle$$

$f(x) = 0$ leaves the sign unchanged. $f(x) = 1$ reverses it.

AMPLITUDES BEFORE THE ORACLE

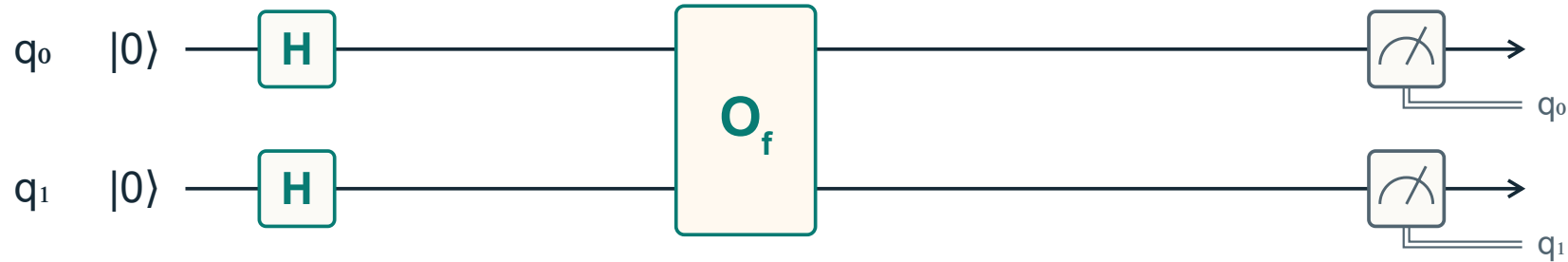


AMPLITUDES AFTER THE ORACLE



The candidate labels stay in place. The amplitude of 10 changes sign.

Does the phase mark make 10 more likely to appear?



AMPLITUDES AFTER THE PHASE MARK



$$P(10) = \left| -\frac{1}{2} \right|^2 = \frac{1}{4}$$

All four probabilities are still $\frac{1}{4}$.

Changing the sign leaves the squared magnitude unchanged.

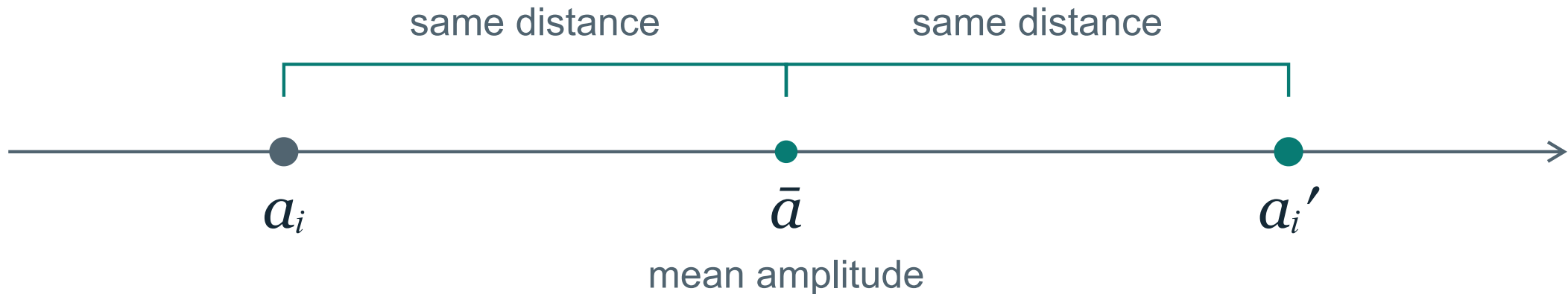
Reflection places an amplitude equally far across the mean.

MEAN OF THE FOUR AMPLITUDES

$$\bar{a} = \frac{a_0 + a_1 + a_2 + a_3}{4}$$

REFLECTED AMPLITUDE

$$a'_i = 2\bar{a} - a_i$$



The **diffuser**, D, applies this rule to each amplitude.

The mean is taken over signed amplitudes.

The diffuser is one fixed matrix.

BASIS ORDER: 00, 01, 10, 11

$$D = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

ONE ROW OF THE MATRIX MULTIPLICATION

$$\begin{aligned} a'_i &= \frac{a_0 + a_1 + a_2 + a_3}{2} - a_i \\ &= 2\bar{a} - a_i \end{aligned}$$

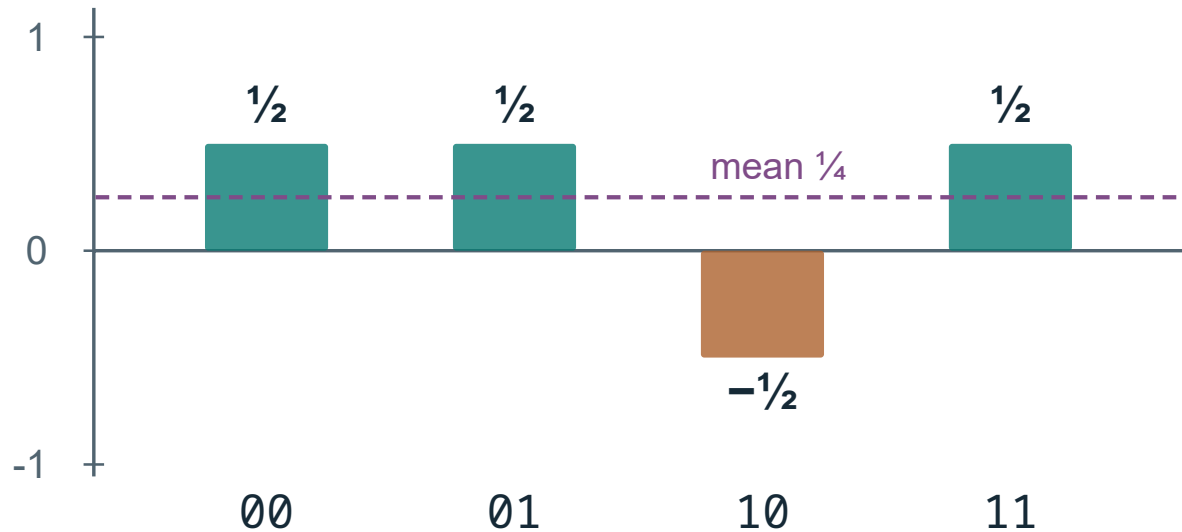
The same D works whichever candidate is marked.

$D^T D = I$: the operation is unitary and preserves normalization.

D acts on the qubits. The mean describes what this fixed matrix does.

After our phase mark, the mean amplitude is $\frac{1}{4}$.

AMPLITUDES AFTER THE ORACLE



MEAN AMPLITUDE

$$\begin{aligned}\bar{a} &= \frac{\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2}}{4} \\ &= \frac{1}{4}\end{aligned}$$

The sign change lowers the mean from $\frac{1}{2}$ to $\frac{1}{4}$.

The reflection cancels three amplitudes and reinforces one.

Candidate	After the oracle	After D: $a'_i = 2(\frac{1}{4}) - a_i$
00	$\frac{1}{2}$	$2(\frac{1}{4}) - \frac{1}{2} = 0$
01	$\frac{1}{2}$	$2(\frac{1}{4}) - \frac{1}{2} = 0$
10	$-\frac{1}{2}$	$2(\frac{1}{4}) - (-\frac{1}{2}) = 1$
11	$\frac{1}{2}$	$2(\frac{1}{4}) - \frac{1}{2} = 0$

$$\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)^T \xrightarrow{D} (0, 0, 1, 0)^T$$

The squared magnitudes still sum to one.

One iteration gives 10 with certainty in this example.



STATE BEFORE MEASUREMENT

$|10\rangle$

MEASUREMENT PREDICTION

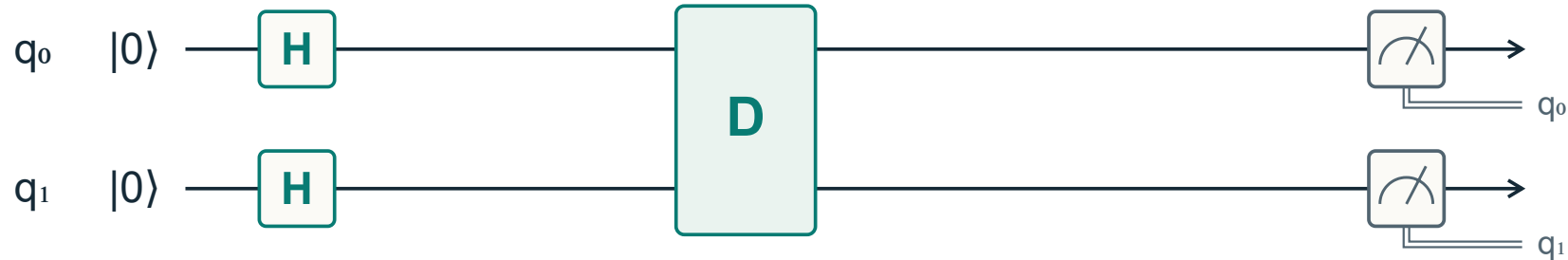
$$P(10) = 1$$

$q_0 = 1, q_1 = 0$. The other candidates have probability zero.

This is amplitude amplification: the accepted outcome becomes more likely.

¹¹ Certainty here is specific to four candidates, one solution and one ideal iteration.

Would the diffuser alone give the same improvement?



The mean stays $\frac{1}{2}$.

$$a'_i = 2\left(\frac{1}{2}\right) - \frac{1}{2} = \frac{1}{2}$$

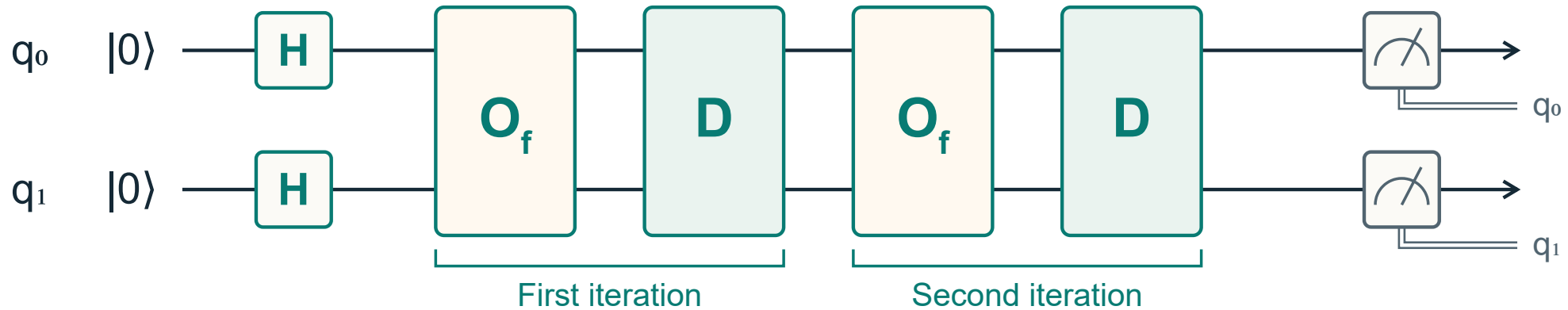
Every amplitude is unchanged.

After uniform preparation	P(10)
Oracle only	$\frac{1}{4}$
Diffuser only	$\frac{1}{4}$
Oracle, then diffuser	1

The phase mark and the diffuser have different jobs. This amplification needs both.

Does a second iteration keep the result certain?

We keep the state from the first iteration and apply the pair again.



$$P(10) \text{ after two iterations} = \frac{1}{4}$$

Another iteration can reduce the probability of success.

The second reflection brings back the other amplitudes.

The first iteration left amplitudes $(0, 0, 1, 0)$.

AFTER THE SECOND ORACLE

$$(0, 0, -1, 0)^T$$

$$\bar{a} = -\frac{1}{4}$$

AFTER THE SECOND DIFFUSER

$$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$$

Unmarked: $2(-\frac{1}{4}) - 0 = -\frac{1}{2}$.

Marked: $2(-\frac{1}{4}) - (-1) = \frac{1}{2}$.

Number of iterations

P(10)

0

1

2

$\frac{1}{4}$

1

$\frac{1}{4}$

We choose the iteration count before the final measurement.

The two reflections give a rotation.

Marked direction: $|w\rangle = |10\rangle$.

Other direction: the normalized equal combination of the remaining candidates.

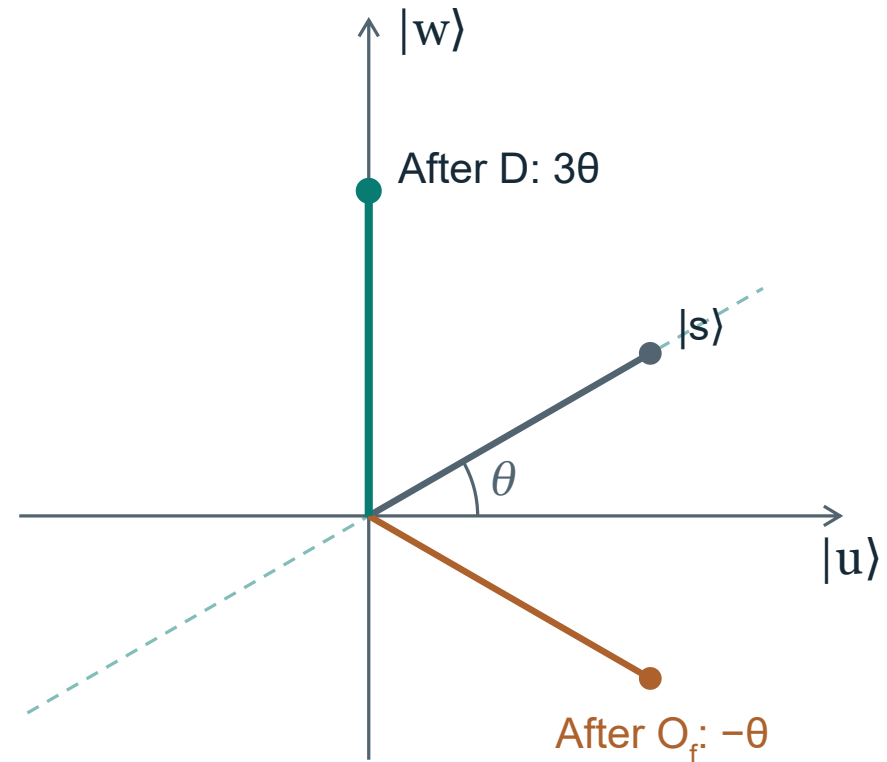
$$|u\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}$$

The unmarked amplitudes stay equal, so they share one direction.

O_f reflects across $|u\rangle$: $\theta \rightarrow -\theta$.

D reflects across $|s\rangle$: $-\theta \rightarrow 3\theta$.

Here $\theta = 30^\circ$. One iteration adds 2θ and reaches 90° , along the marked direction.



A smaller starting amplitude requires more iterations.

With one solution among N candidates, its starting amplitude is $1/\sqrt{N}$.

EACH ITERATION ADDS THE SAME ANGLE

$$\sin \theta = \frac{1}{\sqrt{N}}$$

$$(2r + 1)\theta \approx \frac{\pi}{2} \Rightarrow r \approx \frac{\pi}{4\theta} - \frac{1}{2}$$

r is the number of iterations.

For large N , $\theta \approx 1/\sqrt{N}$.

AN IDEAL EXAMPLE

$N = 1,024$

25 iterations

Success probability

≈ 0.99946

Classical unstructured search

Order N calls

Grover search

Order $\sqrt{N}$ calls

An oracle call contains gates whose cost still counts.

For the 25-iteration example, suppose each call uses these gate counts:

ORACLE

1,000

two-qubit gates per call

DIFFUSER

200

two-qubit gates per call

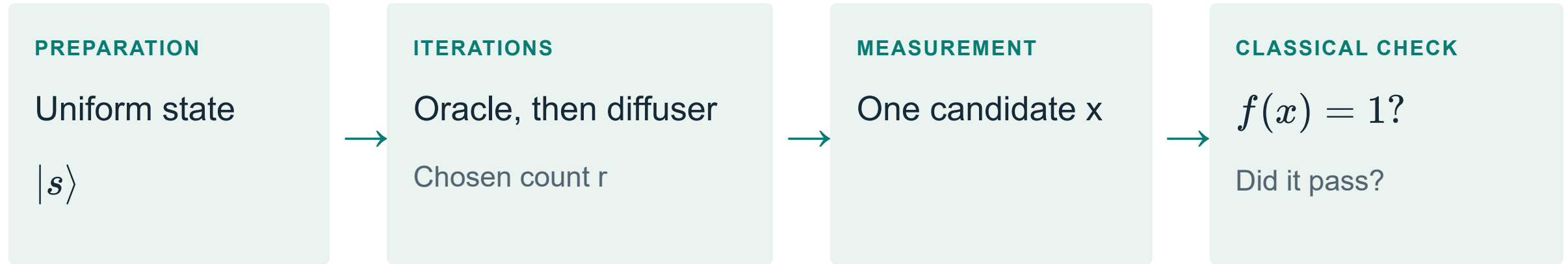
$$25 (1,000 + 200) = 30,000$$

Two-qubit gates in the iteration loop.

These are supplied assumptions for a cost calculation.

Preparation, readout and checking the measured candidate add work.

A comparison includes the complete run.



Access to required data.

Gates and readout.

Fresh attempts if needed.

We compare the same problem, including any useful classical structure.

The full cost determines whether the query saving helps an implementation.

Phase marking, interference and repetition have distinct roles.

- 01 A phase mark preserves probabilities in the computational basis.
 - 02 Interference can make a marked solution more likely to appear.
 - 03 For one solution, Grover needs order $\sqrt{N}$ calls to the checking rule.
-

Grover search uses interference to find a solution with fewer calls to a checking rule.

What happens to the amplitudes in this example?

06.01 · One complete Grover iteration ↗

01

What is the mean after the phase mark?

02

What is the marked amplitude after the diffuser?

03

Which candidate will we measure with certainty?

Let's work through 06.01 together.

We enter the three answers in order, separated by commas.
Leading zeros are part of a bit string.

Candidate order: q_0q_1 . Amplitude order: 00, 01, 10, 11.