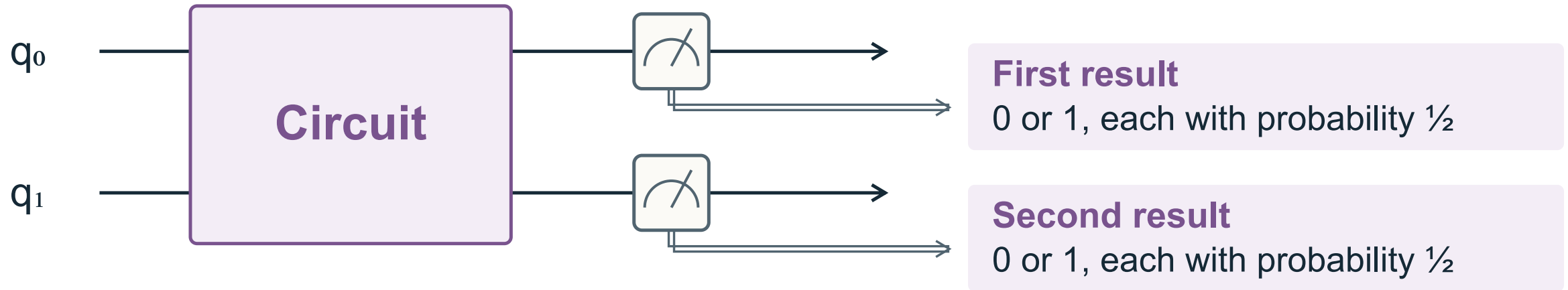


Joint states and entanglement

We can now follow one qubit through a circuit.
What changes when we use two?

We run a circuit with two qubits.

We measure both at the end. Each run starts with the same preparation.



Do these probabilities tell us how often the two results agree on the same run?

No. These individual probabilities allow all three:

Always agree

Always disagree

Agree half the time

To calculate agreement, we need to describe the pair.

Knowing what each qubit does on its own is not always enough to predict what they do together.

- 01 Two qubits share one state with four amplitudes.
- 02 Controlled gates can create entanglement from a product state.
- 03 Entanglement supplies useful correlations without allowing signals faster than light.

Two qubits have four joint amplitudes.

We write $|q_0q_1\rangle$: **q₀** is the left entry.

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

$$\begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix}$$

Entry order: 00, 01, 10, 11.

$$|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$$

Use the same measurement rule:

$$P(01) = |c_{01}|^2$$

n qubits: 2^n amplitudes. One readout: one string of n bits.

Product states multiply the individual amplitudes.

For column vectors, the tensor product is the Kronecker product.

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

00

 ac

01

 ad

10

 bc

11

 bd

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

The two individual vectors determine the four joint entries.

A superposition can still be a product state.

PLUS ON Q_0 · ZERO ON Q_1

$$|+\rangle \otimes |0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

Entangled: the joint state vector cannot be factored into one state vector for each qubit.

Our plus tensor zero example still factors.

Gates acting separately preserve the product form.



$$(H \otimes I)(|0\rangle \otimes |0\rangle) = (H|0\rangle) \otimes |0\rangle = |+\rangle \otimes |0\rangle$$

I leaves the second factor unchanged.

Apply H to q_1 as well:

$$|+\rangle \otimes |0\rangle \longrightarrow |+\rangle \otimes |+\rangle$$

It is still a product.

Can more single-qubit gates create entanglement from a product input?

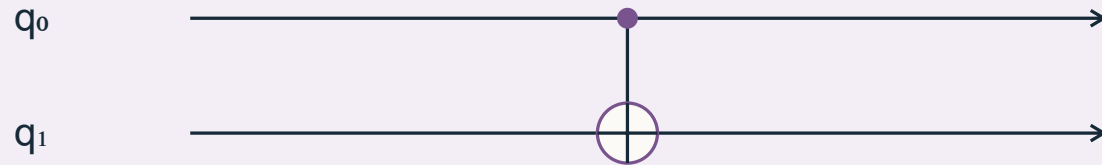
Start with one state vector for each qubit.

Apply any sequence of gates on individual wires.

No. Each step changes a factor and leaves a product.

To create entanglement from this input, we need an operation involving both qubits.

CNOT flips the target label when the control label is 1.



Dot: control q_0

Circled plus: target q_1

$$|00\rangle \mapsto |00\rangle$$

$$|01\rangle \mapsto |01\rangle$$

$$|10\rangle \mapsto |11\rangle$$

$$|11\rangle \mapsto |10\rangle$$

Apply the rule to every term and keep its amplitude.

One reversible operation. CNOT does not measure the control.

H followed by CNOT prepares a Bell state.



AFTER H ON Q_0

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

The input to CNOT.

CNOT, TERM BY TERM

$$|00\rangle \mapsto |00\rangle$$

$$|10\rangle \mapsto |11\rangle$$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\Phi^+\rangle \quad \text{Bell phi plus}$$

Does CNOT copy a plus input?

The target starts in zero. Our calculation gives:

$$\text{CNOT}(|+\rangle \otimes |0\rangle) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

No. Two copies of plus would require:

$$|+\rangle \otimes |+\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Four nonzero amplitudes. The Bell state has two.

Can one state vector for each qubit still describe our Bell pair?

The Bell state cannot factor into two individual vectors.

For a product state, both amplitude products must agree:

$$c_{00}c_{11} = (ac)(bd) = abcd = (ad)(bc) = c_{01}c_{10}$$

BELL STATE / 00 × 11 AMPLITUDES

$$c_{00}c_{11} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2}$$

BELL STATE / 01 × 10 AMPLITUDES

$$c_{01}c_{10} = 0 \cdot 0 = 0$$

$1/2 \neq 0$. No product of two individual vectors gives these amplitudes.

Does CNOT always create entanglement?

$$|+\rangle \otimes |+\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Amplitude	00	01	10	11
Before CNOT	1/2	1/2	1/2	1/2
After CNOT	1/2	1/2	1/2	1/2

CNOT exchanges the 10 and 11 terms. Their amplitudes are equal.

$$\text{CNOT}|++\rangle = |++\rangle$$

The state remains a product. The input matters.

The direct readout gives 00 or 11.

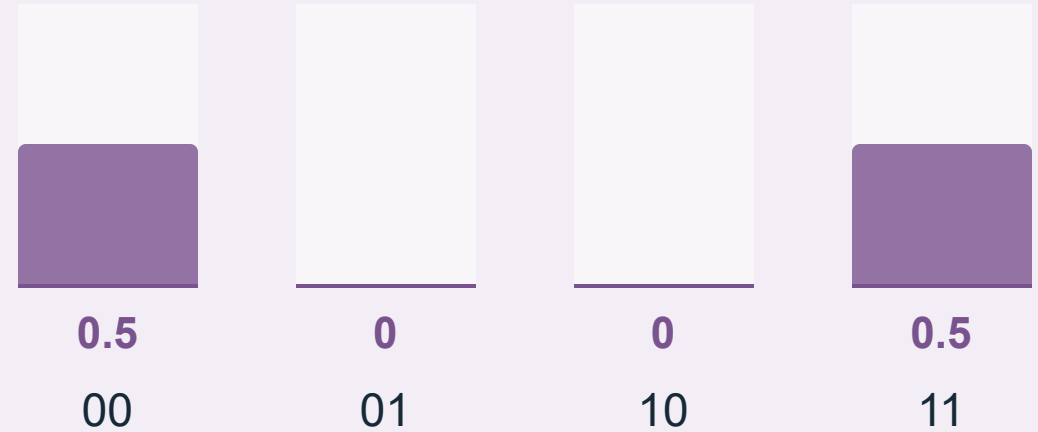
Measure both qubits in the computational basis.

JOINT AMPLITUDES

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

SQUARED MAGNITUDES / PROBABILITIES



The two disagreeing outcomes have zero amplitude.

Each result is uncertain. Their agreement is certain.

$$(P(00), P(01), P(10), P(11)) = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

ONE QUBIT AT A TIME

$$P(q_0 = 0) = P(00) + P(01) = \frac{1}{2}$$

$$P(q_1 = 0) = P(00) + P(10) = \frac{1}{2}$$

Each qubit gives 0 or 1 with equal probability.

THE RESULTS AGREE

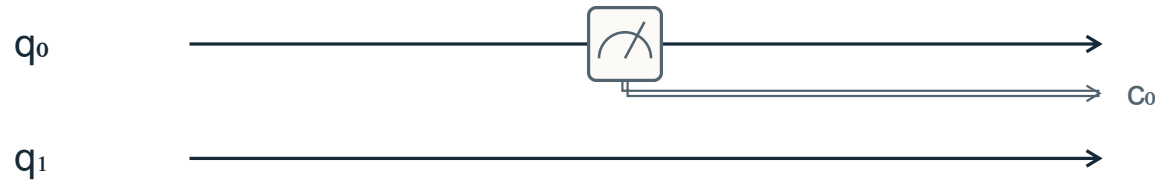
$$P(q_0 = q_1) = P(00) + P(11)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Both recorded labels are the same.

The next zero-or-one readout on q_1 agrees with q_0 's result.

Start with the Bell pair. Measure only q_0 .



Q_0 RECORDS 0 / PROBABILITY $\frac{1}{2}$

State afterwards: $|00\rangle$

The next q_1 readout gives 0.

Q_0 RECORDS 1 / PROBABILITY $\frac{1}{2}$

State afterwards: $|11\rangle$

The next q_1 readout gives 1.

Use the same computational-basis readout, with no intervening gate.

Could a random preparation explain the same results?

PREPARE THE BELL STATE

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

One superposition.

Direct readout: 00 or 11, each with probability $\frac{1}{2}$.

RANDOMLY CHOOSE A PRODUCT PREPARATION

Prepare $|00\rangle$ with probability $\frac{1}{2}$.

Prepare $|11\rangle$ with probability $\frac{1}{2}$.

Choose one on each run.

Direct readout: 00 or 11, each with probability $\frac{1}{2}$.

The same direct measurement probabilities. We need another readout to distinguish them.

Apply H to both qubits before measuring.

INPUT / THE BELL STATE

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



$$(H \otimes H)|\Phi^+\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$$

Follow the joint amplitudes before calculating the readout probabilities.

The unequal-label amplitudes cancel.

$$(H \otimes H)|\Phi^+\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$$

Contribution after H on both	00	01	10	11
From $ 00\rangle/\sqrt{2}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
From $ 11\rangle/\sqrt{2}$	$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
Add the amplitudes	$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\Phi^+\rangle$$

$$P(q_0 = q_1) = 1$$

For the random preparation, average probabilities.

Follow each of the two possible preparations separately.

CHOOSE 00 / PROBABILITY $\frac{1}{2}$

$$|00\rangle \xrightarrow{H \text{ on both}} |++\rangle$$

All four amplitudes are $1/2$.

Each outcome has probability $1/4$.

CHOOSE 11 / PROBABILITY $\frac{1}{2}$

$$|11\rangle \xrightarrow{H \text{ on both}} |--\rangle$$

The amplitudes have different signs.

Each outcome still has probability $1/4$.

For each recorded outcome:

$$\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$$

$$P(q_0 = q_1) = P(00) + P(11) = 1/4 + 1/4 = 1/2.$$

The preparation and the readout both matter.

Preparation	Direct readout $P(q_0 = q_1)$	H on both, then readout $P(q_0 = q_1)$
Bell phi plus	1	1
Randomly prepare 00 or 11, equally	1	1/2

Add amplitudes within the Bell superposition.

Average probabilities across the randomly selected preparations.

Measure q_0 of the Bell pair in 0/1 and ignore its result:

$|00\rangle$ with probability 1/2 or $|11\rangle$ with probability 1/2

For the later H-on-both readout, this gives the random-preparation row.

Compare repeated preparations. One matching pair can come from either procedure.

Alice’s choices leave Bob’s own probabilities unchanged.

Alice holds q_0 . Bob holds q_1 . Each case starts with a fresh Bell pair.

Alice’s action	Joint 0/1 results	Bob’s $P(0)$	Bob’s $P(1)$
Do nothing	00 or 11	$1/2$	$1/2$
Apply X	01 or 10	$1/2$	$1/2$
Measure 0/1 Keep the record	Matching results	$1/2$	$1/2$

Alice’s result predicts Bob’s. She cannot choose her random result.

Can Bob's counts tell him whether Alice chose to measure?

Bob has his own measurement records.

Alice has not sent him hers.

No. Both choices give him the same local probabilities.

Alice's local operations leave those probabilities unchanged when Bob includes all runs.

Alice sends her records → Bob can select runs where she got 0.

That calculation uses information Alice has sent him.

Superdense coding uses a shared pair and a transmitted qubit.

PREVIOUSLY SHARED

1

Bell pair

THEN TRANSMITTED

1

qubit

MESSAGE RECEIVED

2

classical bits

The transmitted qubit is part of the procedure.

The shared entanglement alone does not send the message.

Module 7 develops the encoding and decoding circuit.

Knowing what each qubit does on its own is not always enough to predict what they do together.

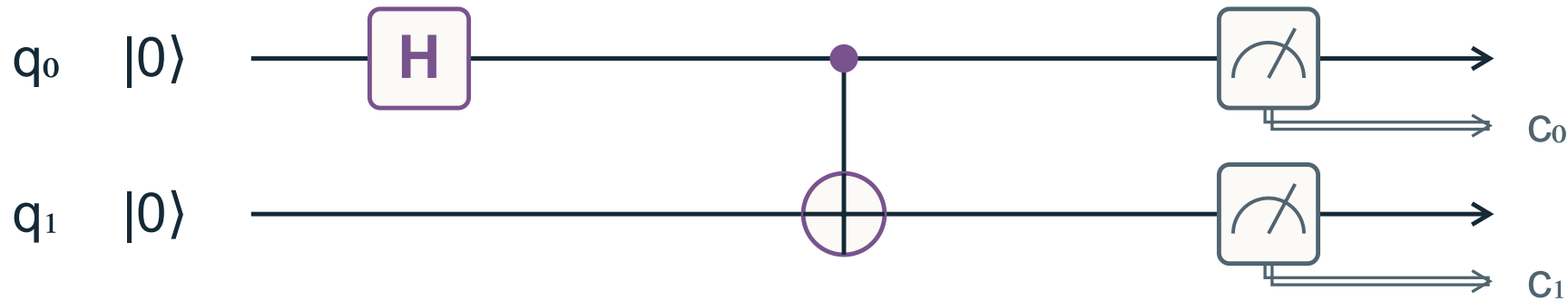
- 01 Two qubits share one state with four amplitudes.

- 02 Controlled gates can create entanglement from a product state.

- 03 Entanglement supplies useful correlations without allowing signals faster than light.

What is the probability that q_0 and q_1 agree?

Start in $|00\rangle$. Measure both qubits in the computational basis.



$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

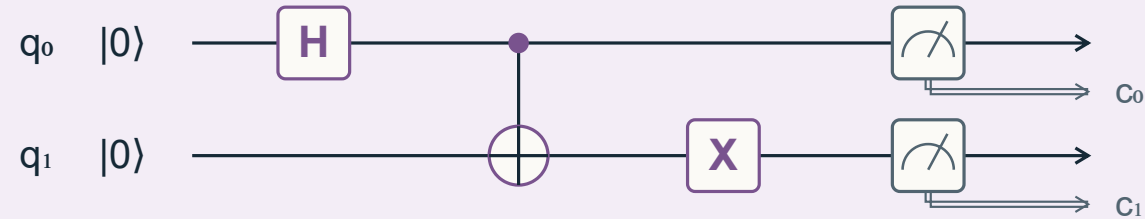
$$P(q_0 = q_1) = |c_{00}|^2 + |c_{11}|^2 = \frac{1}{2} + \frac{1}{2} = \boxed{1.00}$$

Equal labels means 00 or 11. Add those two probabilities.

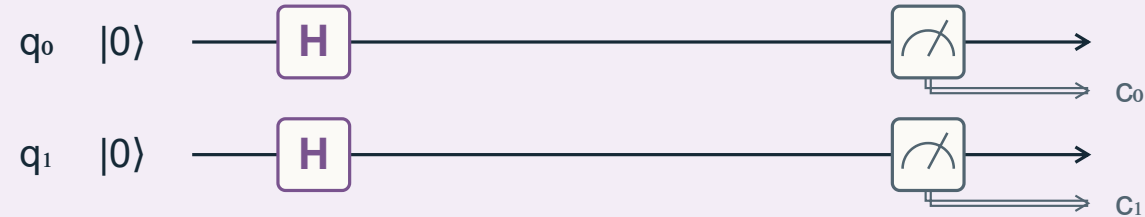
Find $P(q_0 = q_1)$ for each circuit.

Start in $|00\rangle$. q_0 is the top wire and the left ket entry.

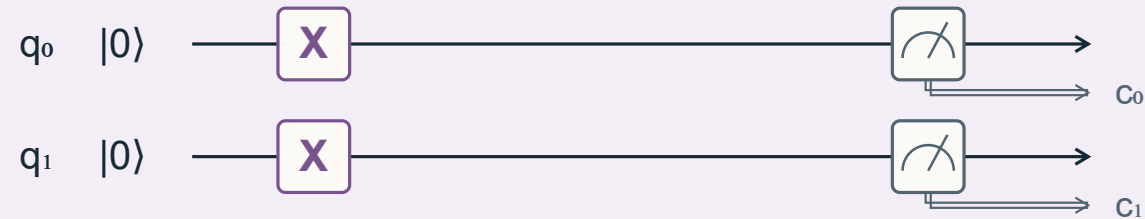
1



2



3



Submit three decimal probabilities. Explain which final states are entangled.