

Is the function constant or balanced?

Our function takes a two-bit input and returns one bit.

Input	Constant zero	Constant one	Balanced example
00	0	1	0
01	0	1	0
10	0	1	1
11	0	1	1

Constant: the same value on every input.

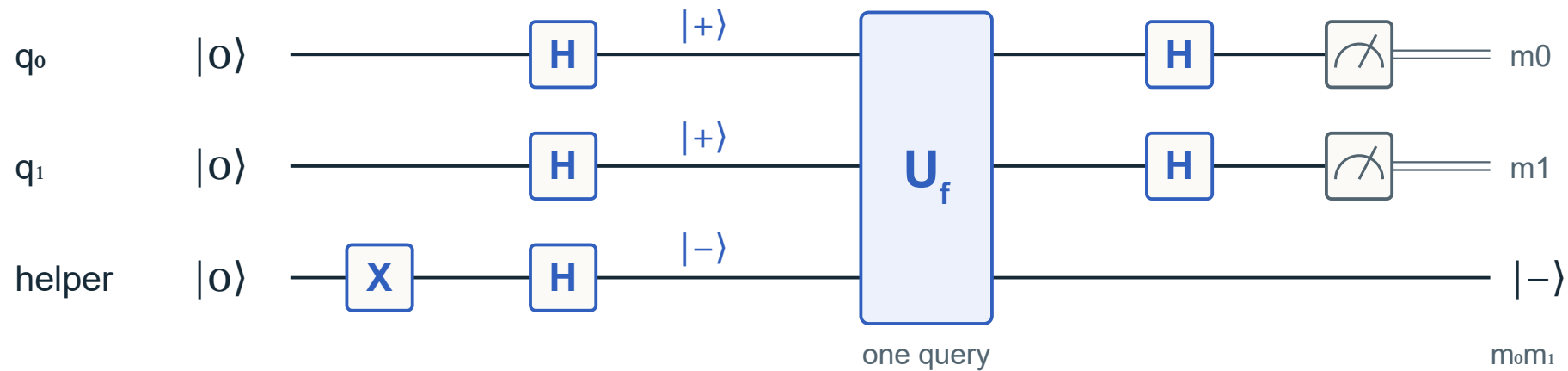
Balanced: zero on half the inputs, one on the other half.

We are promised one of these cases.
How many calls do we need to tell which?

¹ Deutsch–Jozsa problem · These are examples; the supplied function is unknown.

Interference distinguishes constant from balanced functions with one quantum query.

One query = one use of the supplied function operation.



00 → **constant** · Any other input pair → balanced, under the promise.

Oracles

Phase kickback

Interference

Oracles let us count function calls independently of their implementation.

An **oracle** is an operation we treat as a black box.

The box can contain a circuit with several gates.

In our example, the oracle encodes f as a quantum operation.

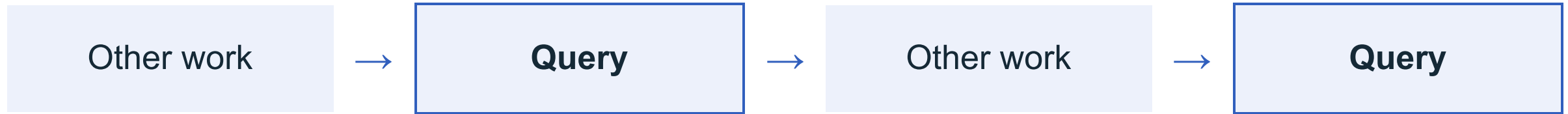
CLASSICAL CALL

The function value for one input.

QUANTUM CALL

A gate that also acts on superpositions of inputs.

Counting queries isolates how much function access an algorithm needs.



Two queries.

The full running time also includes the work inside the oracle and in the rest of the circuit.

Two matching answers can leave this four-input problem undecided.

TWO QUERIES SO FAR

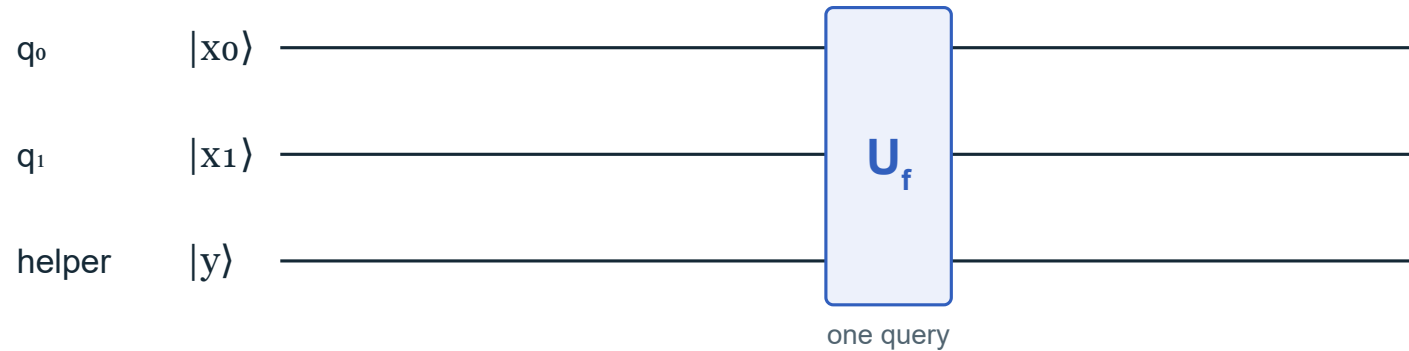
Input	Known value
00	0
01	0
10	?
11	?

BOTH COMPLETIONS FIT

Input	Constant	Balanced
00	0	0
01	0	0
10	0	1
11	0	1

A third distinct query settles it: another zero → constant; a one → balanced.
Three queries in the exact classical worst case.

Our oracle keeps the input labels and updates a helper.



$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle.$$

x : the two-bit input label.

y : the helper's basis label, 0 or 1.

$|x\rangle|y\rangle$: the joint state.

$\oplus$: XOR, addition modulo two.

The helper is also called the target or ancilla.

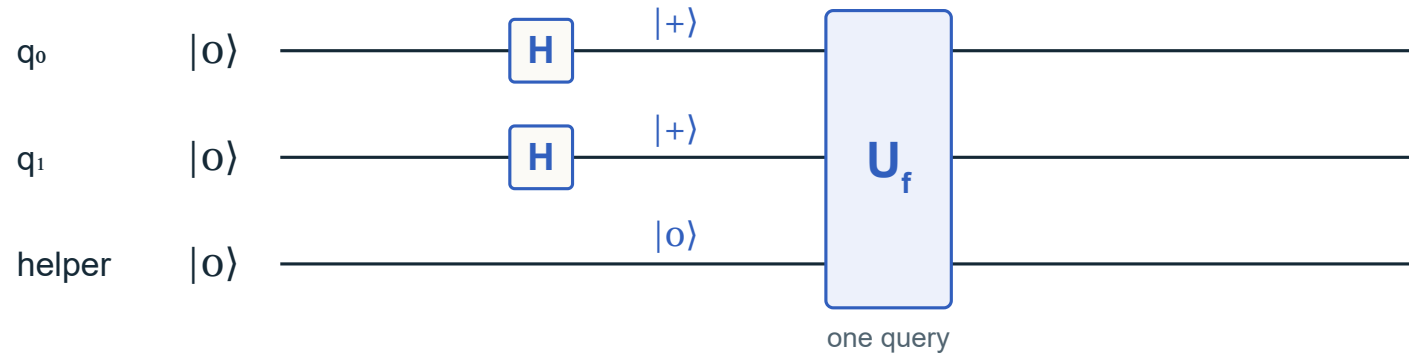
The oracle applies X to the helper when $f(x)$ is one.

Two inputs from our balanced example:

x	$f(x)$	Input to U_f $ x\rangle y\rangle$	Output $ x\rangle y \oplus f(x)\rangle$
00	0	$ 00\rangle 0\rangle$	$ 00\rangle 0\rangle$
00	0	$ 00\rangle 1\rangle$	$ 00\rangle 1\rangle$
10	1	$ 10\rangle 0\rangle$	$ 10\rangle 1\rangle$
10	1	$ 10\rangle 1\rangle$	$ 10\rangle 0\rangle$

$f(x) = 0$: no flip. $f(x) = 1$: X on the helper.

One query acts coherently on the input superposition.



$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$\left(\frac{1}{2} \sum_x |x\rangle \right) |0\rangle \xrightarrow{U_f} \frac{1}{2} \sum_x |x\rangle |f(x)\rangle.$$

Measuring both registers here gives one input–value pair.

The helper is zero in this comparison. The sum covers 00, 01, 10 and 11.

Flipping a minus helper changes only its overall sign.



$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

$$X|-\rangle = \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -|-\rangle.$$

For the helper alone, this overall sign represents the same physical state.

The query supplies a minus sign exactly where $f(x)$ is one.

$$|x\rangle|-\rangle \xrightarrow{U_f} (-1)^{f(x)}|x\rangle|-\rangle.$$

Input x	$f(x)$	Helper action	Factor on this term
00	0	No flip	+1
01	0	No flip	+1
10	1	X	-1
11	1	X	-1

The sign depends on the input term.

The helper stays minus while the input acquires the signs.

The joint state after the query:

$$\frac{1}{2} (|00\rangle|-\rangle + |01\rangle|-\rangle - |10\rangle|-\rangle - |11\rangle|-\rangle)$$

INPUT

$$\frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2}$$

A new relative-sign pattern.



HELPER

$$|-\rangle$$

The same state.

Phase kickback: the conditional target operation changes phases on the input.

Constant functions change all signs together.

Input label	Constant zero	Constant one	Balanced example
00	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
01	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
10	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
11	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

An overall sign leaves the physical state unchanged. The balanced case changes relative signs.

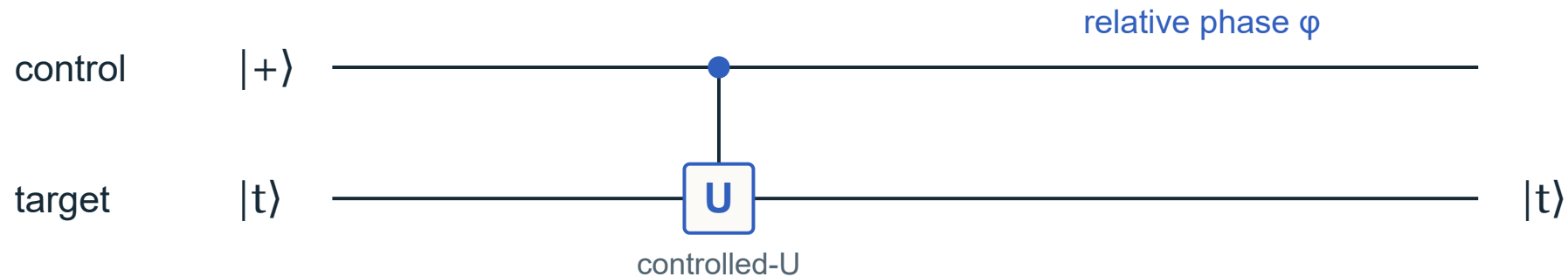
$$\underbrace{O_f|x\rangle = (-1)^{f(x)}|x\rangle}$$

phase oracle: input action with the minus helper

A target phase can become a relative phase on a control.

Suppose U leaves our chosen target state unchanged apart from a phase:

$$U|t\rangle = e^{i\phi}|t\rangle.$$

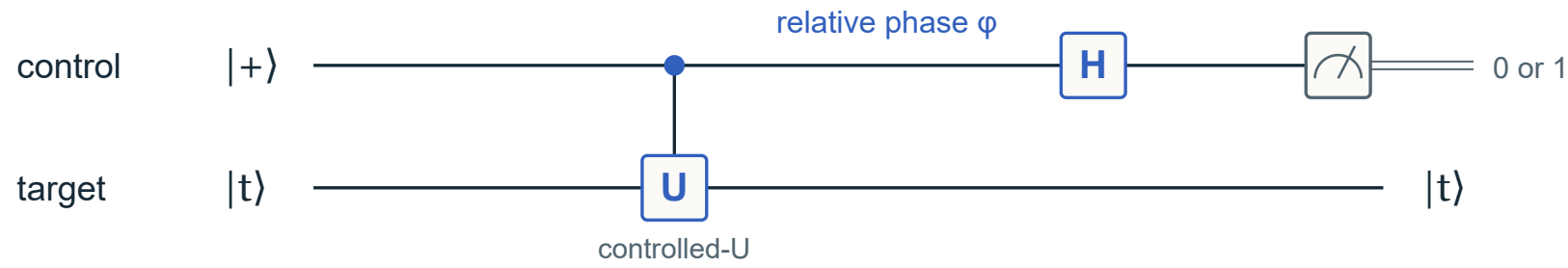


$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}|t\rangle \longrightarrow \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}|t\rangle.$$

The target stays in the same state. The control acquires a relative phase.

$|t\rangle$ names the target state; the target may contain several qubits.

Interference on the control lets us learn about that phase.



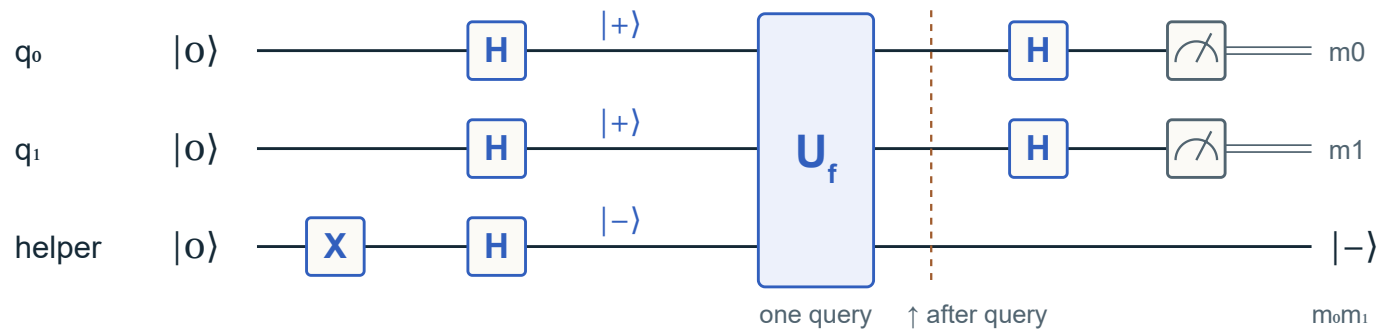
$$\frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}} \xrightarrow{H} \frac{1 + e^{i\phi}}{2}|0\rangle + \frac{1 - e^{i\phi}}{2}|1\rangle.$$

Phase from U	Control after kickback	P(0) after H	P(1) after H
$\phi = 0$	$(0\rangle + 1\rangle)/\sqrt{2}$	1	0
$\phi = \pi$	$(0\rangle - 1\rangle)/\sqrt{2}$	0	1
$\phi = \pi/2$	$(0\rangle + i 1\rangle)/\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Phase estimation learns about U through measurements on control qubits.

A single measurement generally does not determine the phase.

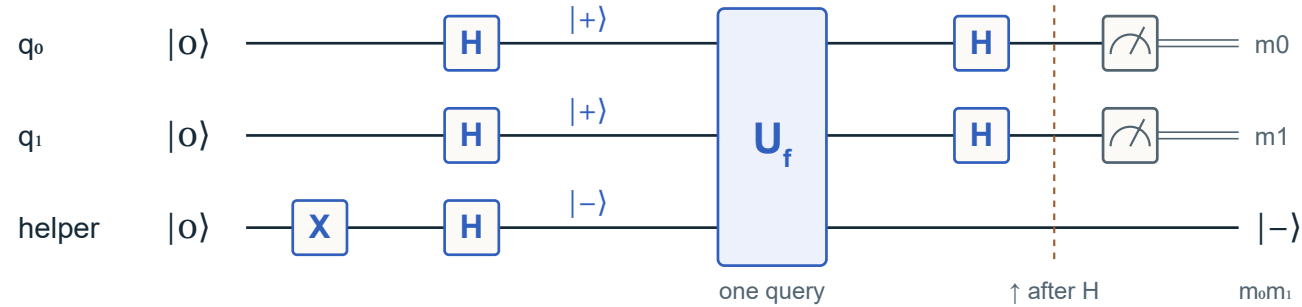
The signs leave immediate measurement probabilities unchanged.



Input label	Constant-zero amplitude	Balanced-example amplitude	Probability in either case
00	$+1/2$	$+1/2$	$1/4$
01	$+1/2$	$+1/2$	$1/4$
10	$+1/2$	$-1/2$	$1/4$
11	$+1/2$	$-1/2$	$1/4$

Measuring here cannot distinguish constant from balanced.

The final Hadamards distinguish our two examples.



For each input qubit: $H|+\rangle = |0\rangle$, $H|-\rangle = |1\rangle$.

CONSTANT ZERO

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = |+\rangle|+\rangle \xrightarrow{H \otimes H} |00\rangle$$

OUR BALANCED EXAMPLE

$$\frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2} = |-\rangle|+\rangle \xrightarrow{H \otimes H} |10\rangle$$

10 is the result for this balanced example. Other balanced functions can give other results.

Each input term contributes half its amplitude to 00.

Can another balanced function still give 00?

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Both coefficients of $|0\rangle$ are $+1/\sqrt{2}$. For two qubits, they multiply to $+1/2$.

FOR EXAMPLE, THE BASIS TERM 10

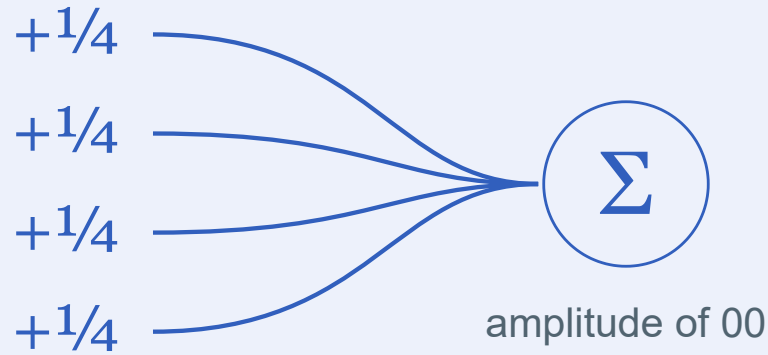
$$\begin{aligned} (H \otimes H)|10\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle. \end{aligned}$$

$$\underbrace{-\frac{1}{2}}_{\text{input amplitude}} \times \underbrace{\frac{1}{2}}_{\text{factor to 00}} = \underbrace{-\frac{1}{4}}_{\text{contribution to 00}}$$

We add all contributions to 00, then square the magnitude.

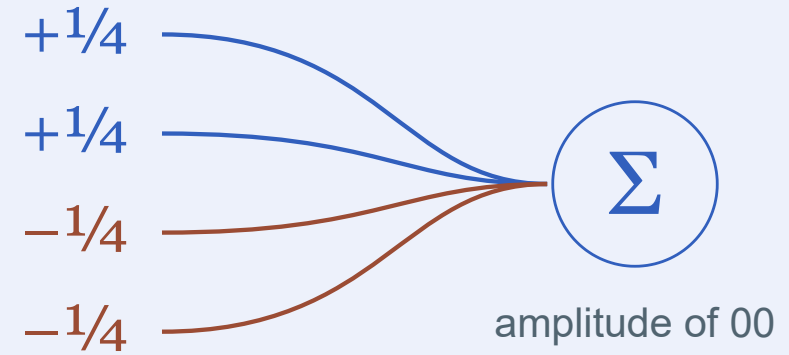
Equal signs reinforce; opposite signs cancel.

CONSTANT ZERO



$$A(00) = 1, \quad P(00) = 1.$$

OUR BALANCED EXAMPLE

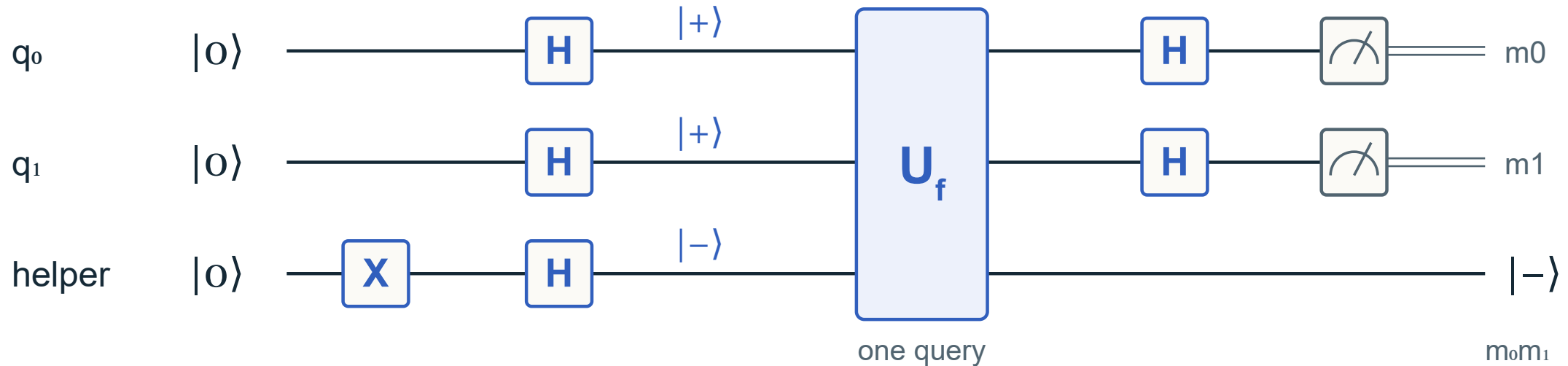


$$A(00) = 0, \quad P(00) = 0.$$

Each contribution is $\pm 1/4$. Constant one gives $A(00) = -1$ and also $P(00) = 1$.

The measured input distinguishes constant from balanced.

Equal amplitudes $\rightarrow$ function-dependent signs $\rightarrow$ interference



00 $\rightarrow$ constant

01, 10 or 11 $\rightarrow$ balanced

One function query. An exact classification under the promise, with ideal operations.

The query comparison depends on the required certainty.

The same constant-or-balanced promise, with N possible inputs:

Exact classification	Classical worst case	Quantum
Four inputs	3 queries	1 query
N inputs	$N/2 + 1$ queries	1 query

$N = 2^n$ for n input bits.

If a small error probability is allowed, classical random sampling can use only a few queries.

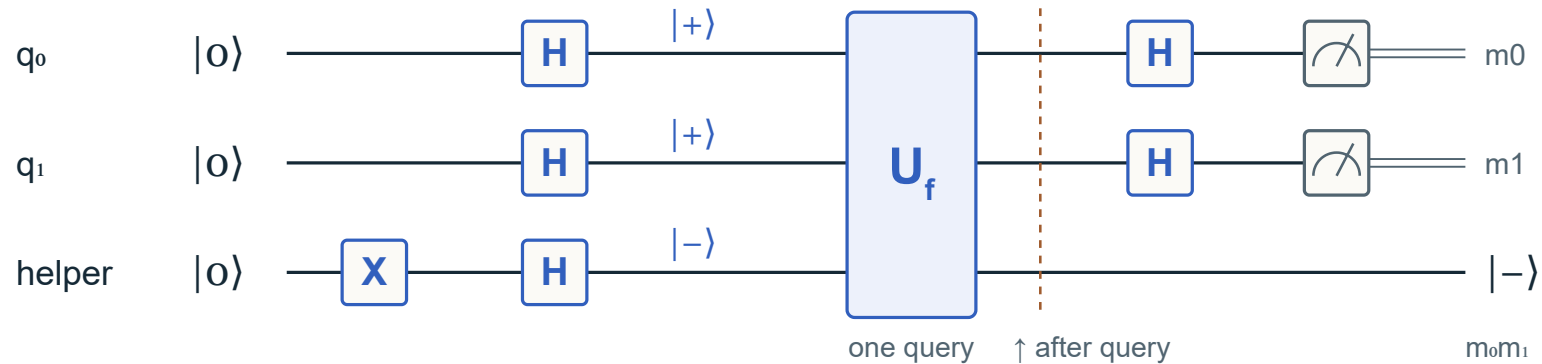
Query count is one part of the cost. Implementing the oracle and the surrounding gates also takes work.

Interference distinguishes constant from balanced functions with one quantum query.

- 01 Oracles let us analyse algorithms by counting queries.
- 02 Phase kickback turns conditional target flips into phases on the input.
- 03 Interference turns the function's phase pattern into a measurable answer.

PRACTICE · A DIFFERENT PROMISED FUNCTION

What does the same circuit tell us about this function?



Input x	f(x)
00	0
01	1
10	1
11	0

What signs does the query produce?

What is $P(00)$ after the final H gates?

What would measuring immediately after the query tell us?